

A Note on Coalgebras and Presheaves

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Abstract

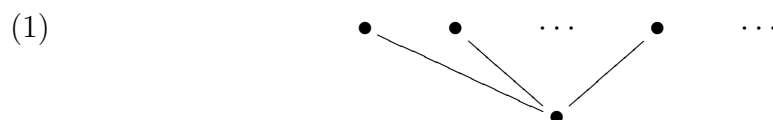
We show that the category of coalgebras of a wide-pullback preserving endofunctor on a category of presheaves is itself a category of presheaves. This illustrates a connection between Jacobs' temporal logic of coalgebras and Ghilardi and Meloni's presheaf semantics for modal logic.

1 Introduction

Recall that a presheaf category is one which is equivalent to a functor category $[\mathcal{C}^{op}, \mathbf{Set}]$ for some small category \mathcal{C} . We show that the category of coalgebras of a wide-pullback preserving endofunctor T on a presheaf category is itself a presheaf category. In fact, we construct a freely generated path category \mathcal{C} from the functor T such that T -coalgebras correspond to presheaves on \mathcal{C} . This construction is an adaptation of one used by Carboni and Johnstone [1] in showing that the category obtained by Artin gluing along a limit preserving functor between presheaf categories is also a presheaf category.

2 Wide Pullbacks

Definition 2.1 (*cf. [1]*) A wide pullback is the limit of a diagram indexed by a poset (X, \leq) with a greatest element \top and such that $x \leq y$ iff $y = \top$ for all $x, y \in X$ (see below). \square



One important class of wide-pullback preserving functors $\mathbf{Set} \rightarrow \mathbf{Set}$ are the so-called *partial product functors* [1,5]. Polynomial functors on \mathbf{Set} (built from the identity, constant functors, sums, products and composition) are all partial product functors. Also, if $T: \mathbf{Set} \rightarrow \mathbf{Set}$ is a partial product functor,

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then the functor mapping a set X to the cofree T -coalgebra over X is a partial product functor, as is the functor mapping X to the free T -algebra over X (see [5, Lemma 2.4]).

Example 2.2 Let $T: \mathbf{Set} \rightarrow \mathbf{Set}$ be the subfunctor of the exponential functor $(-)^{\mathbb{N}}$ consisting of the ‘eventually constant functions’. More precisely,

$$TX = \{f \in X^{\mathbb{N}} : (\exists m)(\forall n \geq m)(\forall n' \geq m) f(n) = f(n')\}.$$

It is routine to verify that T preserves pullbacks (indeed it preserves all finite limits). However, it does not preserve the infinite product $P = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots$ since there is no eventually constant map $\mathbb{N} \rightarrow P$ which corresponds to the tuple $(f_n: \mathbb{N} \rightarrow \mathbb{N})_{n \in \mathbb{N}}$ where $f_n(x) = \min(x, n)$. Thus T does not preserve wide pullbacks. \square

Proposition 2.3 If \mathcal{A} is a complete category, then every wide-pullback preserving endofunctor $T: \mathcal{A} \rightarrow \mathcal{A}$ has a final coalgebra.

Proof. A wide-pullback preserving functor whose domain category is complete preserves all connected limits [1, Lemma 2.1]. It follows that T preserves limits indexed by the chain ω^{op} . Thus the final coalgebra of T may be constructed as the limit of the ω^{op} -chain $1 \leftarrow T1 \leftarrow T^2 1 \leftarrow \dots$ in the standard manner. \square

Let \mathcal{A} be a complete category, suppose $T: \mathcal{A} \rightarrow \mathcal{A}$ preserves wide pullbacks, and let $\alpha: A \rightarrow TA$ be given. We define a ‘reduction’ of T to a limit preserving endofunctor on the slice category \mathcal{A}/A as follows. T has an obvious lifting to a functor $T_A: \mathcal{A}/A \rightarrow \mathcal{A}/TA$, and composing this with the pullback functor $\alpha^*: \mathcal{A}/TA \rightarrow \mathcal{A}/A$ we obtain an endofunctor $T_\alpha: \mathcal{A}/A \rightarrow \mathcal{A}/A$. Thus, for an object $f: B \rightarrow A$ of \mathcal{A}/A , $T_\alpha f$ is defined by the pullback below.

$$(2) \quad \begin{array}{ccc} \bullet & \xrightarrow{\quad} & TB \\ T_\alpha f \downarrow & & \downarrow Tf \\ A & \xrightarrow{\quad \alpha \quad} & TA \end{array}$$

Proposition 2.4 (i) T_α preserves all (small) limits.

(ii) $\text{Coalg } T_\alpha$ is isomorphic to the slice category $\text{Coalg } T/(A, \alpha)$.

Proof. (i) Observing that wide pullbacks in a slice category \mathcal{A}/A are created by the forgetful functor $\mathcal{A}/A \rightarrow \mathcal{A}$, it is easy to see that they are preserved by T_A . Furthermore, T_A clearly preserves final objects. Thus T_A preserves all small limits, since any small limit may be constructed from final objects and wide pullbacks. The functor α^* is a right adjoint, and thus preserves all limits. It follows that $T_\alpha = \alpha^* \cdot T_A$ is continuous.

(ii) If $f: B \rightarrow A$ is a map in \mathcal{A} , then a coalgebra structure $f \rightarrow T_\alpha f$ clearly corresponds to a map $\beta: B \rightarrow TB$ such that f is a coalgebra map $(B, \beta) \rightarrow (A, \alpha)$. This extends to an isomorphism of categories acting as identity on homsets. \square

3 Coalgebras as Presheaves

3.1 Bimodules and Presheaves

We first recall from [6,1] the definition of the bicategory² of small categories and bimodules, and its equivalent presentation as the 2-category \mathbf{PreSh} of presheaf categories and continuous functors.

Definition 3.1 A bimodule (also called a profunctor or distributor) from a category \mathcal{A} to a category \mathcal{B} , written $\phi : \mathcal{A} \multimap \mathcal{B}$, is a functor

$$\phi : \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \mathbf{Set}.$$

We write $\mathbf{Mod}(\mathcal{A}, \mathcal{B})$ for the category of bimodules $\mathcal{A} \multimap \mathcal{B}$ and natural transformations between them. One can define a composition of bimodules; then small categories, bimodules, and natural transformations form a bicategory \mathbf{Mod} . \square

A bimodule $\phi : \mathcal{A} \multimap \mathcal{B}$ may be regarded as a functor $\mathcal{A} \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$, but the category of such functors is equivalent to the category of cocontinuous functors $[\mathcal{A}^{\text{op}}, \mathbf{Set}] \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$: the two components of the equivalence being, respectively, restriction and left Kan extension along the Yoneda embedding $y_{\mathcal{A}} : \mathcal{A} \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}]$. Furthermore, there is a 1–1 correspondence between cocontinuous functors $[\mathcal{A}^{\text{op}}, \mathbf{Set}] \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$ and continuous functors $[\mathcal{B}^{\text{op}}, \mathbf{Set}] \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}]$ (map a functor to its right adjoint). Thus, for each pair of small categories \mathcal{A} and \mathcal{B} , there is an equivalence between $\mathbf{Mod}(\mathcal{A}, \mathcal{B})$ and the category of continuous functors $[\mathcal{B}^{\text{op}}, \mathbf{Set}] \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}]$. In fact, this extends to a biequivalence of bicategories from \mathbf{Mod} to \mathbf{PreSh} . We don't need to verify this last fact, but we do need an explicit calculation of the image of the bimodule ϕ under the above equivalence, which we denote $[\phi, -]^{\mathcal{B}}$, cf. [1].

Given a bimodule $\phi : \mathcal{A} \multimap \mathcal{B}$, writing $\bar{\phi}$ for the functor $\lambda a \lambda b \phi(b, a) : \mathcal{A} \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$, consider $\text{Lan}_{y_{\mathcal{A}}} \bar{\phi}$, the left Kan extension of $\bar{\phi}$ along the Yoneda embedding $y_{\mathcal{A}} : \mathcal{A} \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}]$. This is a cocontinuous functor which is given by the formula

$$\text{Lan}_{y_{\mathcal{A}}} \bar{\phi}(P) = \text{Colim}(\text{Elts}(P) \xrightarrow{U} \mathcal{A} \xrightarrow{\bar{\phi}} [\mathcal{B}^{\text{op}}, \mathbf{Set}])$$

for a presheaf $P : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Set}$, where $\text{Elts}(P)$ is the comma category $(1 \downarrow P)$.

Fixing a presheaf $Q : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Set}$, a morphism $\text{Lan}_{y_{\mathcal{A}}} \bar{\phi}(P) \Rightarrow Q$ corresponds to a cocone from the diagram $\bar{\phi} \cdot U$ to Q . The data for such a cocone is, for each pair $(a, x) \in \text{Elts}(P)$, a choice of a natural transformation $\alpha_{(a,x)} : \bar{\phi}(a) \Rightarrow Q$ —this choice being natural in (a, x) . This amounts to a natural transformation $P \Rightarrow [\phi, Q]^{\mathcal{B}}$, where the functor

$$[\phi, -]^{\mathcal{B}} : [\mathcal{B}^{\text{op}}, \mathbf{Set}] \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}]$$

is defined by

$$[\phi, Q]^{\mathcal{B}}(a) = [\mathcal{B}^{\text{op}}, \mathbf{Set}](\phi(-, a), Q),$$

² It is convenient to use the language of bicategories here, but no knowledge of them is required to read this section.

and is by this definition right adjoint to $\text{Lan}_{\mathcal{Y}\mathcal{A}}\overline{\phi}$.

3.2 The Main Construction

Suppose \mathcal{A} is a small category and $\phi : \mathcal{A} \nrightarrow \mathcal{A}$. Let $\mathcal{G}(\phi)$ be the graph with

- (i) nodes: the set of objects of \mathcal{A} ;
- (ii) edges: for each arrow $f : a \rightarrow b$ of \mathcal{A} an edge $f : a \rightarrow b$ of $\mathcal{G}(\phi)$, and, for each pair of objects a, b of \mathcal{A} and each $e \in \phi(b, a)$, an edge $e : b \rightarrow a$ of $\mathcal{G}(\phi)$.

From the graph $\mathcal{G}(\phi)$ we freely generate a category, which we denote $\mathcal{C}(\phi)$, subject to the following equations on composition in $\mathcal{C}(\phi)$ (written as $\cdot_{\mathcal{C}(\phi)}$).

- (a) For composable morphisms f, g of \mathcal{A} , $f \cdot_{\mathcal{C}(\phi)} g = f \cdot g$;
- (b) if $e \in \phi(b, a)$ and $f : b' \rightarrow b$ is an arrow of \mathcal{A} , then $e \cdot_{\mathcal{C}(\phi)} f = \phi(f, a)e$;
- (c) if $e \in \phi(b, a)$ and $f : a \rightarrow a'$ is an arrow of \mathcal{A} , then $f \cdot_{\mathcal{C}(\phi)} e = \phi(b, f)e$.

A graph homomorphism $P : \mathcal{G}(\phi)^{op} \rightarrow \mathbf{Set}$ consists of a graph homomorphism $P_0 : \mathcal{A}^{op} \rightarrow \mathbf{Set}$ plus a family of mappings, indexed over pairs of objects $a, b \in \mathcal{A}$,

$$\alpha(b, a) : \phi(b, a) \rightarrow P_0(b)^{P_0(a)}.$$

By exponential transposition this last datum amounts to a family of mappings,

$$\overline{\alpha}(b, a) : P_0(a) \rightarrow P_0(b)^{\phi(b, a)}.$$

The graph homomorphism P will be a functor if it preserves identities in $\mathcal{C}(\phi)$ and the three types of composition (a)-(c) above. Preservation of identities and composites of type (a) is equivalent to P_0 being a functor $\mathcal{A}^{op} \rightarrow \mathbf{Set}$. Given this, P preserves composites of type (b) precisely when, for each $x \in P_0(a)$, $\overline{\alpha}(-, a)x$ is a natural transformation $\phi(-, a) \Rightarrow P_0$, i.e.,

$$\overline{\alpha}(-, a) : P_0(a) \rightarrow [\phi, P_0]^{\mathcal{A}}(a).$$

In addition, preservation of composites of type (c) is the same as requiring that the above family of maps is natural in $a \in \mathcal{A}$. Thus we have shown that a presheaf P on $\mathcal{C}(\phi)$ amounts to a pair $(P_0, \overline{\alpha})$, where P_0 is a presheaf on \mathcal{A} and

$$\overline{\alpha} : P_0 \rightarrow [\phi, P_0]^{\mathcal{A}}$$

is a natural transformation.

Let us suppose we have another presheaf Q on $\mathcal{C}(\phi)$, consisting of a presheaf Q_0 on \mathcal{A} and a family of maps $\beta(b, a) : \phi(b, a) \rightarrow Q_0(b)^{Q_0(a)}$, indexed by objects $b, a \in \mathcal{A}$. A natural transformation $\Xi : P \Rightarrow Q$ is precisely a natural transformation $\Xi_0 : P_0 \Rightarrow Q_0$ such that the left hand diagram, below, commutes for each pair of objects $a, b \in \mathcal{A}$. But, by exponential transposition, this is just the same as requiring that the right hand diagram commutes.

$$\begin{array}{ccc}
 \phi(b, a) & \xrightarrow{\alpha(b, a)} & P_0(b)^{P_0(a)} \\
 \beta(b, a) \downarrow & & \downarrow \Xi_{0, b}^{P_0(a)} \\
 Q_0(b)^{Q_0(a)} & \xrightarrow{Q_0(b)^{\Xi_{0, a}}} & Q_0(b)^{P_0(a)}
 \end{array}
 \qquad
 \begin{array}{ccc}
 P_0 & \xrightarrow{\bar{\alpha}} & [\phi, P_0]^{\mathcal{A}} \\
 \Xi_0 \downarrow & & \downarrow [\phi, \Xi_0]^{\mathcal{A}} \\
 Q_0 & \xrightarrow{\bar{\beta}} & [\phi, Q_0]^{\mathcal{A}}
 \end{array}$$

It is now clear that there is an isomorphism of categories between $[\mathcal{C}(\phi)^{op}, \mathbf{Set}]$ and the category of coalgebras of $[\phi, -]^{\mathcal{A}}$. Since any continuous endofunctor $T: [\mathcal{A}^{op}, \mathbf{Set}] \rightarrow [\mathcal{A}^{op}, \mathbf{Set}]$ is of the form $[\phi, -]^{\mathcal{A}}$, for some ϕ , we obtain the following result.

Theorem 3.2 *If T is a continuous endofunctor on a presheaf category, then $\mathbf{Coalg} T$ is itself a presheaf category.*

Corollary 3.3 *If T is a wide-pullback preserving endofunctor on a presheaf category, then $\mathbf{Coalg} T$ is itself a presheaf category.*

Proof. Proposition 2.3 tells us that there is a final T -coalgebra (A, α) . From Proposition 2.4 it follows that $\mathbf{Coalg} T \cong \mathbf{Coalg} T / (A, \alpha) \cong \mathbf{Coalg} T_\alpha$. Since T_α is continuous, and the property of being a presheaf category is preserved by taking slices [4, Corollary 2.18], the result follows. \square

Remark 3.4 *Our proof followed an idea of Carboni and Johnstone [1] who showed that Artin gluing along a continuous functor between presheaf categories yields again a presheaf category; now we can explain the precise relationship. Given an endofunctor $T: \mathcal{B} \rightarrow \mathcal{B}$, $\mathbf{Coalg} T$ has a universal property in the 2-category \mathbf{CAT} of large categories, functors and natural transformations: it is the oplax limit of a diagram with shape*

$$(3) \quad \bullet \curvearrowright$$

where the node is labelled \mathcal{B} and the edge T .

From Theorem 3.2 it follows that, if $\phi: \mathcal{A} \nrightarrow \mathcal{A}$ is a bimodule, then $\mathcal{C}(\phi)$ has a similar universal property in \mathbf{Mod} – it is the oplax limit of a diagram whose shape is given in (3), but where the node is labelled by \mathcal{A} and the edge by ϕ . On the other hand, [1] constructs the collage of a bimodule $\phi: \mathcal{A} \nrightarrow \mathcal{B}$. This is the oplax limit in \mathbf{Mod} of a diagram of shape $\bullet \rightarrow \bullet$. But this diagram corresponds to Artin gluing in \mathbf{CAT} .

4 Conclusion and Related Work

Ghilardi and Meloni [2] consider an interpretation of modal logic based on presheaves rather than Kripke models. Specifically they consider a temporal logic with two modal operators, respectively interpreted as *generated sub-presheaf* and *cogenerated sub-presheaf*.

Let $P: \mathcal{C}^{op} \rightarrow \mathbf{Set}$ be a presheaf on a small category \mathcal{C} . A predicate ϕ on P is a family of sets $\phi(C)$, indexed by the set of objects \mathcal{C}_0 of \mathcal{C} , such that $\phi(C) \subseteq P(C)$ for all $C \in \mathcal{C}_0$. For each predicate ϕ we have the co-generated

sub-presheaf $\underline{\phi}$, i.e., the maximum sub-presheaf of P contained in ϕ . This is given by the formula

$$\underline{\phi}(C) = \{x : (\forall B \in \mathcal{C}_0)(\forall f: B \rightarrow C) \ P(f)(x) \in \phi(B)\}.$$

We also have the generated sub-presheaf $\overline{\phi}$, i.e. the minimum sub-presheaf of P containing ϕ . This is given by the formula

$$\overline{\phi}(C) = \{y : (\exists B \in \mathcal{C}_0)(\exists g: C \rightarrow B)(\exists x \in \phi(B)) \ y = P(g)(x)\}.$$

Suppose T is a wide-pullback preserving set functor and \mathcal{C} is the category constructed in Section 3 such that $\text{Coalg } T$ and $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ are isomorphic. Examining the details of this isomorphism we find that if a presheaf P corresponds to a coalgebra (A, α) , then predicates ϕ on P are in 1-1 correspondence with predicates (i.e., subsets) S of A . Under this correspondence, the generated sub-presheaf $\overline{\phi}$ becomes the smallest sub-coalgebra of (A, α) containing S , and the co-generated sub-presheaf $\underline{\phi}$ becomes the largest sub-coalgebra of (A, α) contained in S . Going in the other direction, Jacobs [3] has shown how to represent any given category of presheaves as a category of coalgebras such that generated and cogenerated sub-presheaves agree with generated and co-generated sub-coalgebras. In view of these connections it would be of interest to compare Jacobs' coalgebraic semantics for modal logic with the analysis of Ghilardi and Meloni.

It is possible to generalize the ideas of Ghilardi and Meloni to sheaves on a site. That is, for a predicate ϕ on a sheaf P we have a generated subsheaf $\overline{\phi}$ and a co-generated subsheaf $\underline{\phi}$. We would like to see if these correspond to generated and cogenerated sub-coalgebras under the coalgebras-as-sheaves correspondence presented in [5] for coalgebras of weak-pullback preserving functors. In general, a Grothendieck topos is equivalent to a category of sheaves on many different sites, and it seems to us that the key to solving this problem is to find the 'right' sites for the toposes considered in [5].

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